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Minimum implicational basis for ∧-semidistributive lattices

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Received 13 January 2006; received in revised form 21 February 2006; accepted 21 April 2006

Communicated by L. Boasson

Abstract

For a $\wedge$-semidistributive lattice L, we study some particular implicational systems and show that the cardinality of a minimum implicational basis is polynomial in the size of join-irreducible elements of the lattice L. We also provide a polynomial time algorithm to compute a minimum implicational basis for L.

Keywords: Algorithms; Lattice; Closure system; Minimum implicational basis

1. Introduction

This paper deals with the computation of a minimum implicational basis for a closure system. Computing a minimum implicational basis for a lattice given by its poset of irreducible elements is an important problem, which has applications to many areas of computer science, in particular to databases and AI [1,4,6,7,10]. For a survey on this problem and related areas, see [3].

The complexity of this problem remains open for general lattices. Recent progress on the status of this problem, and in particular solvability by limited non-determinism [5], suggests however that this problem is more likely to be expected tractable than intractable [4].

It has been already shown that this problem is tractable for the two classes of locally distributive lattice [2] and of modular lattices [14]. In this paper we show by using a dependence relation in [11] that the class of $\wedge$-semidistributive lattices is another tractable case.

Consider a finite set U. A subset $\mathcal{C} \subseteq 2^U$ is said to be a closure system if $\mathcal{C}$ is closed under set-intersection and containing the set U. An implication on U is an ordered pair (A, B) of subsets of U, denoted by $A \to B$. The set A is called the premise and the set B the conclusion of the implication $A \to B$. Let Σ be a set of implications on U. A subset $D \subseteq U$ is Σ-closed if for each implication $A \to B$ in Σ, $A \subseteq D$ implies $B \subseteq D$. The set of Σ-closed subsets of U, denoted by $\mathcal{C}(\Sigma)$, is a closure system on U. Conversely, given a closure system $\mathcal{C}$ on U, a family Σ of implications on U is said an implicational basis for $\mathcal{C}$ if $\mathcal{C} = \mathcal{C}(\Sigma)$. An implicational basis is said minimum if it has a minimum number of implications.

In this paper, we study the latticial version of this problem. We view a lattice L as the closure system $\mathcal{C}_L$ on the set $J(L)$ of its join-irreducible elements. More precisely, put $J(a) = \{j \in J(L)\colon\ j \leqslant a\}$ for $a \in L$.

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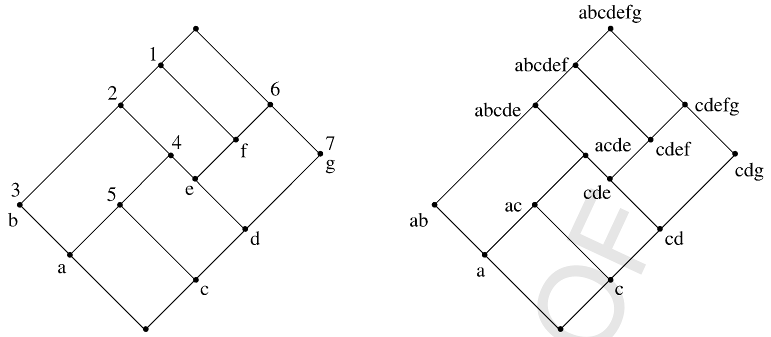
0020-0190/$ – see front matter © 2006 Published by Elsevier B.V.
doi:10.1016/j.ipl.2006.04.004



(b) (b)

Fig. 1. (a) A lattice L where join-irreducible (resp. meet-irreducible) elements are labeled by letters (resp. numbers); (b) The closure system cl associated to L.

Then $\mathcal{C}_L = \{J(a)\colon a \in L\}$ is a closure system on $J(L)$ which, as a lattice ordered by inclusion, is isomorphic to L.

Fig. 1 gives an example of the closure system $\mathcal{C}_L$ associated to a lattice L.

The closure system $\mathcal{C}_L$ can be defined by the set of its meet-irreducible elements $\mathcal{M}(\mathcal{C}_L) = \{J(m)\colon m \in M(L)\}$, where $M(L)$ denotes the set of meet-irreducible elements of L. Each element of $\mathcal{C}_L$ can be obtained as intersection of some elements of $\mathcal{M}(\mathcal{C}_L)$.

The problem we study is:

Problem: Minimum implicational basis
Instance: The set of meet-irreducible elements $\mathcal{M}(\mathcal{C}_L)$ of the closure system $\mathcal{C}_L$.
Question: Find a minimum basis Σ for $\mathcal{C}_L$.

This problem remains open for general lattices. Duquenne [2] has given a latticial version of this problem and shown that it is polynomial for upper locally distributive lattices or antimatroid. Recently, Wild [14] has proposed a polynomial time algorithm to compute an optimal[1] implicational basis for modular lattices. In the following, we study the case of $\wedge$-semidistributive lattices. For such lattices we show that the number of implications of a minimum implicational basis is at most $|J(L)|^2$ and give a polynomial time algorithm to compute such a basis.

2. Some properties of $\wedge$-semidistributive lattices

Let L be a finite lattice. We note $\vee$ the join operation, $\wedge$ the meet operation and $\prec$ the cover relation of L. If j is a join-irreducible element of L, we use j_* to denote the unique element covered by j. Dually, we use m^* to denote the unique element covering a meet-irreducible element m.

We will use the arrow relations introduced by Wille [15]: for $x, y \in L$, $x \downarrow y$ means that x is a minimal element of $\{z \in L\colon z \nleq x\}$, $x \uparrow y$ means that y is a maximal element of $\{z \in L\colon z \ngeq y\}$ and $x \updownarrow y$ means that $x \uparrow y$ and $x \downarrow y$. Recall that $\uparrow, \downarrow, \updownarrow$ are relations defined on $J(L) \times M(L)$, where $J(L)$ is the set of join-irreducible elements and $M(L)$ the set of meet-irreducible elements of L.

In the following, we deal essentially with $\wedge$-semidistributive lattices. Let us recall that a lattice L is said $\wedge$-semidistributive if for all elements $x, y, z \in L$, $x \wedge y = x \wedge z$ implies $x \wedge y = x \wedge (y \vee z)$. A $\wedge$-semidistributive lattice is said semidistributive if for all elements x, y, z, $x \vee y = x \vee z$ implies $x \vee y = x \vee (y \wedge z)$. The following characterization of these lattices are well known (see, for example, [6]):

Property 1. *A finite lattice L is $\wedge$-semidistributive if and only if for any $j \in J(L)$ there exists a unique $m \in M(L)$ such that $j \updownarrow m$.*

For any $\wedge$-semidistributive lattice L and $j \in J(L)$, we denote by $m(j)$ the unique element $m \in M(L)$ such that $j \updownarrow m$.

We define the mapping $\gamma : J(L) \to 2^{M(L)}$ by $\gamma(j) = \{m \in M(L)\colon j \downarrow m\}$. This mapping was introduced in [12] to define colored posets, which provides a new representation for lattices, and specially for upper locally distributive lattices. Fig. 2 shows the γ mapping of the lattice of Fig. 1. Note that this lattice is semidistributive.

We consider one of the standard dependence relations defined on join-irreducible elements (assuming

[1] An implication is known as optimal if the sum of the cardinality of the premises and the conclusions of all the implications is minimal.

	1	2	3	4	5	6	7	γ
a						$\updownarrow$	$\downarrow$	$\{6,7\}$
b				$\updownarrow$	$\downarrow$	$\uparrow$		$\{4,5\}$
c			$\updownarrow$					$\{3\}$
d			$\uparrow$		$\updownarrow$			$\{5\}$
e			$\uparrow$		$\uparrow$		$\updownarrow$	$\{7\}$
f		$\updownarrow$		$\downarrow$			$\uparrow$	$\{2,4\}$
g	$\updownarrow$	$\downarrow$		$\downarrow$				$\{1,2,4\}$

Fig. 2. The arrow relations and mapping γ of the lattice in Fig. 1.

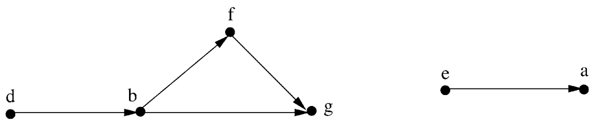


Fig. 3. The graph of the B relation for the lattice in Fig. 1.

that the lattice L is $\wedge$-semidistributive) as follows (see, for example, [8,11]):

Let $j, j' \in J(L)$.
Then $j\,B\,j'$ iff $j \neq j'$, $j' \nleqslant m(j)$, $j'_* \leqslant m(j)$.

For an illustration of that definition, see Fig. 3.

There are relationships between the existence of cycles in the graph of the relation B and some classes of lattices. For example, Nation has shown that a $\wedge$-semidistributive lattice is semidistributive if and only if it contains no B-cycle of length 2 [11].

The following lemma gives a rewriting of the definition of the relation B using the mapping γ.

Lemma 1. *Let L be a $\wedge$-semidistributive lattice, $j, j' \in J(L)$.*

$j\,B\,j'$ iff $j \neq j'$ and $m(j) \in \gamma(j')$.

3. Minimum implicational basis for $\wedge$-semidistributive lattices

In this section, we give a polynomial time algorithm to compute a minimum implicational basis for a $\wedge$-semidistributive lattice.

We start with two technical lemmas on closed sets of a closure system $\mathcal{C}_L$. The first one is obvious since the elements of $\mathcal{C}_L$ are order ideals of the induced poset by $J(L)$.

Lemma 2. *Let $j, j' \in J(L)$ such that $j < j'$ and $X \in \mathcal{C}_L$. Then $j' \in X$ implies $j \in X$.*

Consider now a $\wedge$-semidistributive lattice L and $j, j' \in J(L)$ such that jBj'. We denote by $P_{jj'}$ the set $J(j_*) \cup J(j')$.

Lemma 3. *Let L be a $\wedge$-semidistributive lattice and $j, j' \in J(L)$ such that jBj' and $X \in \mathcal{C}_L$. Then $P_{jj'} \subseteq X$ implies $j \in X$.*

Proof. Let $x \in L$ such that $X = J(x)$ and $P_{jj'} \subseteq X$. Since $J(j_*) \subset X$ this implies that $j_* \vee j' \leqslant x$, and then it suffices to prove that $j \leqslant j_* \vee j'$.

Suppose that $j \nleqslant j_* \vee j'$ and let $m' \in M(L)$ be a maximal element of $\{z \in L \mid z \ngeqslant j$ and $z \geqslant j_* \vee j'\}$. By definition of m', we have $j \uparrow m'$. Moreover $j \downarrow m'$ since $j_* \leqslant m'$. Thus $j \updownarrow m'$.

Consider now the meet-irreducible $m(j)$ associated with j. Then $j' \nleqslant m(j)$ since $j\,B\,j'$. Thus since $j' \leqslant m'$, m' and $m(j)$ are two distinct elements such that $j \updownarrow m'$ and $j \updownarrow m(j)$. This contradicts the fact that L is $\wedge$-semidistributive. $\square$

We can now define a particular set of implications associated to a $\wedge$-semidistributive lattice L. Let $\Sigma_1 = \{j \to J(j)\}$, $\Sigma_2 = \{P_{jj'} \to j \mid j' \in J(L)$ and $j\,B\,j'\}$ and $\Sigma = \Sigma_1 \cup \Sigma_2$.

For example, the sets of implications Σ_1 and Σ_2 for the lattice in Fig. 1 are $\Sigma_1 = \{b \to ab, d \to cd, e \to cde, f \to cdef, g \to cdg\}$ and $\Sigma_2 = \{acd \to e, abc \to d, acdef \to b, acdg \to b, cdeg \to f\}$.

The following theorem shows that Σ is an implicational basis for $\mathcal{C}_L$.

Theorem 1. *Let L be a $\wedge$-semidistributive lattice. Then the set of implications Σ is an implicational basis for $\mathcal{C}_L$.*

Proof. We need to show that $\mathcal{C}_\Sigma = \mathcal{C}_L$.

Let $X \in \mathcal{C}_L$. By Lemma 2, X is Σ_1-closed. By Lemma 3, X is Σ_2-closed. Then X is Σ-closed and $\mathcal{C}_L \subseteq \mathcal{C}_\Sigma$.

Now let us show that $\mathcal{C}_\Sigma \subseteq \mathcal{C}_L$. Let $X \in \mathcal{C}_\Sigma$. Let $x_0 = \bigvee X$, i.e., the least closed set containing X. Clearly X is an ideal since it is Σ_1-closed. Suppose that $X \notin \mathcal{C}_L$ and let j be a minimal element of $J(x_0)\backslash X$. Since $j \leqslant x_0$, we have $x_0 \nleqslant m(j)$. Moreover $X \nsubseteq J(m(j))$, otherwise one would have $\bigvee X \leqslant m(j)$ and then $\bigvee X \neq x_0$. Thus there exists an element $j' \in X$ such that $m(j) \in \gamma(j')$ and therefore $P_{jj'} \to j \in \Sigma$ with $P_{jj'} \subseteq X$ and $j \notin X$. Then X is not Σ-closed, which concludes the proof. $\square$

Corollary 1. *Let L be a $\wedge$-semidistributive lattice. Then there exists an implicational basis for $\mathcal{C}_L$ with at most $|B| + |J(L)|$ implications, where $|B|$ is the number of arcs in the relation B.*

Data: Let L be a $\wedge$-semidistributive lattice and $\mathcal{M}(C_L)$ the set of meet-irreducible elements of $\mathcal{C}_L$.
Result: A minimum basis Σ of the closure system C_L.
begin
  $\Sigma = \emptyset$;
  for $j \in J(L)$ **do**
    $\Sigma = \Sigma \cup \{j \to \varphi(j)\}$;
    for $j' \in J(L)$ **do**
      $P = (\varphi(j)) \backslash \{j\} \cup \varphi(j')$;
      $\Sigma = \Sigma \cup \{P \to (P)\}$;
  $\Sigma =$ a nonredundant cover of Σ;
end

Algorithm 1. Minimum-Basis($\mathcal{M}(C_L)$).

Clearly the set Σ of implications obtained as above is in general not minimum. For instance, for the set Σ associated to the lattice in Fig. 1, the implication $acdg \to b$ is redundant[2] and can be removed from Σ without changing $\mathcal{C}(\Sigma)$.

In the following we give a polynomial time algorithm to compute a minimum basis for a $\wedge$-semidistributive lattice.

3.1. Algorithm

This is based on Theorem 1 and the algorithm in [13]. Indeed, the algorithm in [13] computes a minimum basis (called there a minimum cover) from any given basis in polynomial time.

Let $\mathcal{M}(\mathcal{C}_L)$ be the set of meet-irreducible elements. Consider the closure operator $\varphi : 2^J \to 2^J$, with for $X \subseteq J$, $\varphi(X) = \bigcap\{M \in \mathcal{M}(\mathcal{C}_L) \mid X \subseteq M\}$. The images of the mapping φ are said closed sets, and they correspond to the elements of the closure system $\mathcal{C}_L$.

Remark 1. We replaced $P \to j$ by $P \to \varphi(P)$ to guarantee the minimality after the calculation of a nonredundant cover of Σ.

Remark 2. Let us note that Algorithm 1 does not compute the same Σ as that of Theorem 1. This to avoid the computation of the relation B. But like the whole of the implications calculated by Algorithm 1 contains all implications of Theorem 1 (relative with the preceding remark), this guaranteed to us to have a cover of $\mathcal{C}_L$.

Theorem 2. *Let L be a $\wedge$-semidistributive lattice. Then Algorithm* 1 *computes a minimum implicational basis Σ of $\mathcal{C}_L$ in* $\mathrm{O}(|J|^5 + |J|^3|\mathcal{M}(\mathcal{C}_L)|)$ *time complexity. Moreover, the size of Σ is at most $|J(L)|^2$ implications.*

Proof. Theorem 1 guarantees that Σ is a basis for the closure system $\mathcal{C}_L$. Since the conclusions of all implications are closed by the mapping φ, the result in [13] guarantees that a not redundant basis is minimum.

Computing the closure of a set $X \subseteq J(L)$ by φ can be done in $\mathrm{O}(|J(L)||\mathcal{M}(\mathcal{C}_L)|)$ time complexity. Thus the total time complexity for computing a basis is in $\mathrm{O}(|J(L)|^3|\mathcal{M}(\mathcal{C}_L)|)$. Now computing a not redundant basis can be done in $\mathrm{O}(|J(L)||\Sigma|^2)$. Since Σ has at most $|J(L)|^2$ implications, we conclude that the time complexity of Algorithm 1 is in $\mathrm{O}(|J(L)|^5 + |J(L)|^3|\mathcal{M}(\mathcal{C}_L)|)$. $\square$

Uncited references

[9]

Acknowledgements

The authors are very grateful to the referees for very helpful comments.

References

[1] N. Caspard, B. Monjardet, The lattices of closure systems, closure operators, and implicational systems on a finite set: A survey, Discrete Applied Mathematics 127 (2) (2003) 241–269.
[2] V. Duquenne, The core of finite lattice, Discrete Mathematics 88 (1991) 133–147.
[3] T. Eiter, G. Gottlob, Identifying the minimal transversals of a hypergraph and related problems, SIAM Journal on Computing 24 (6) (1995) 1278–1304.
[4] T. Eiter, G. Gottlob, Hypergraph transversal computation and related problems in logic and AI, in: European Conference on Logics in Artificial Intelligence (JELIA'02), 2002, pp. 549–564.
[5] M.L. Fredman, L. Khachiyan, On the complexity of dualization of monotone disjunctive normal forms, Journal of Algorithms (21) (1996) 618–628.
[6] B. Ganter, R. Wille, Formal Concept Analysis: Mathematical Foundations, Springer-Verlag, Berlin, 1996.
[7] J.L. Guigues, V. Duquenne, Families minimales d'implications informatives resultant d'un tableau de donnes binaires, Mathétiques et Sciences humaines 95 (1986) 5–18.
[8] R. Freeze, K. Jezek, J.B. Nation, Free Lattices, American Mathematical Society, Providence, RI, 1995.
[9] P. Janssen, L. Nourine, A simplicial scheme for meet-semidistributive lattices and interval collapsing, Universalis Algebra 50 (2) (2003) 171–178.
[10] D. Maier, The Theory of Relational Data Bases, Computer Science Press, Rockville, MD, 1983.
[11] J.B. Nation, Unbounded semidistributive lattices, Algebra and Logic 39 (2000) 87–92.
[12] L. Nourine, Une structuration algorithmique de la théorie des treillis, Habilitation à diriger des recherches, Université de Montpellier II, France, July 2000.

[2] An implication $A \to B$ in Σ is said redundant in Σ if it can be derived using Armstrong rules from $\Sigma \backslash \{A \to B\}$.

[13] R.C. Shock, Computing the minimum cover of functional dependencies, Information Processing Letters 3 (22) (1986) 157–159.
[14] M. Wild, Optimal implicational bases for finite modular lattices, Quaestiones Mathematicae 23 (2000) 153–161.
[15] R. Wille, Subdirect decomposition of concept lattices, Algebra Universalis 17 (1983) 275–287.